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Algebraic disturbances in stratified shear flows

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Algebraic disturbances, a non-modal component of the linear perturbation fields, are shown to be an essential feature of stratified shear flows. We find that they must be included even in situations where the modes form a complete set, for such completeness does not extend to the space of these ill-behaved functions.

If the Richardson number Ri is less than $\frac{1}{4}$ anywhere in the flow, the algebraic disturbances are very generally instabilities of the system, growing without limit as time $t \rightarrow \infty$.

Both of these results are in direct contradiction with the currently accepted viewpoint. We examine the previous research in this field to locate the source of this discrepancy.

The algebraic instabilities are not form preserving, and display extreme distortion as they evolve. In the asymptotic limit they appear as quasi-horizontal flow fields, with a vertical 'wavelength' that tends to zero. As such, they must be expected to induce secondary shear instabilities and cascade into motions of smaller (horizontal) scale.

1. Introduction

In the many geophysical problems that involve questions about the stability of stratified shear flows, recourse is often made to the conventional linearized hydrodynamic stability theory. This approach examines modes with a time dependence of the form $e^{\sigma t}$. If eigenvalues with a real part of σ greater than zero can be found, the flow is said to be unstable while, if no such σ exists, the flow is declared to be stable against infinitesimal perturbations. This 'ansatz' (as Case (1960*a*) calls it) is acceptable only if all other solutions of the linearized equations decay, or are stationary, in time. To date, the only 'other' solutions that have been identified are continua of singular functions that exist in a flow with non-zero mean shear. Eliassen, Hoiland & Riis (1953) and Case (1960*a*, *b*) laid the foundations for studying these solutions through examination of flows with constant shear profiles. Both sets of authors concluded that perturbations composed of the singular solutions are stable: amplitudes are either stationary or decaying in the long-time limit. As was first pointed out by Miles (1961), these analyses are easily generalized to arbitrary shear flows without a change in the conclusions. Consequently the 'ansatz' is now generally accepted and stability investigations concentrate on the search for exponentially growing modes.

In this paper we re-examine the behaviour of the (so-called) algebraic solutions, and conclude, contrary to the earlier work, that they produce instabilities that grow in time if the local Richardson number Ri falls below $\frac{1}{4}$ anywhere in the flow (but is not everywhere zero). Moreover, such solutions, stable or otherwise, always contribute to shear-flow disturbances. The analysis begins with an examination of flows that have constant shear and constant Brunt Väisälä frequency. The approach used is a somewhat simplified version of Case's (1960*b*) definitive formulation of the initial-value problem. The final stages differ from those presented by Case; here the simplified formulation makes obvious why at one critical step we depart radically from previous conclusions.

The analysis is generalized to arbitrary shear flows, and we conclude that algebraic instabilities should be a general feature of flows with $Ri < \frac{1}{4}$ somewhere in the profile. The explicit initial-value formulation shows why the approach used by Eliassen *et al.* failed to reveal these instabilities. It also shows that the algebraic solutions are in no sense dependent for their existence on any lack of completeness of the normal modes of the system: algebraic solutions are a consequence of the singular nature of the shear-flow equations and must be included in solutions of the initial-value problem even if a 'complete' set of normal modes can also be identified.

Apart from a very special class of profiles (which may not even exist), the necessary and sufficient condition for algebraic instabilities in a stratified shear flow is that Ri should be less than $\frac{1}{4}$ somewhere in the flow (but not everywhere zero).

2. Formulation

The problem is posed as the time development for $t \geqslant 0$ of perturbations imposed (or specified) at time $t = 0$ in the two space dimensions x and z of a given mean flow. The co-ordinate z is aligned with the direction opposite to that of the gravitational field $\mathbf{g}$ and the mean state is homogeneous in x. At all times $t \geqslant 0$ the governing equations are

$$\rho\, D\mathbf{V}/Dt = -\nabla p + \rho \mathbf{g} \quad \text{(Euler's equation)}, \tag{2.1}$$

$$\nabla . \mathbf{V} = 0 \quad \text{(incompressibility)} \tag{2.2}$$

and

$$D\rho/Dt = 0 \quad \text{(mass conservation)}. \tag{2.3}$$

The mean state is given as

$$\bar{\mathbf{V}} = \hat{\mathbf{x}} U(z), \quad \bar{\rho}(z), \quad \bar{p}(z) \quad \text{with} \quad d\bar{p}(z)/dz = -\bar{\rho}(z)\, g.$$

Departures from the mean state

$$\rho_1 = \rho - \bar{\rho}, \quad p_1 = p - \bar{p}, \quad \mathbf{V}_1 = \hat{\mathbf{x}} u + \hat{\mathbf{z}} w = \mathbf{V} - \bar{\mathbf{V}}$$

are assumed to be of sufficiently small amplitude that (2.1)–(2.3) may be linearized in them. At time $t = 0$ these departures are taken to be well-behaved functions ρ_0, p_0, u_0 and w_0 of x and z. The set (2.1)–(2.3) is linearized in the perturbations and a Laplace transform applied to the time dependence. A Fourier transform is then taken over the x co-ordinate and the resulting equations are manipulated to yield a single differential equation in the transforms of w. The Boussinesq approximation is introduced, i.e. all terms involving $d\bar{\rho}/dz$ are dropped unless they are multiplied by the factor g.

Using the transform notation

$$\hat{w}(k,z,s) = \int_{\infty}^{\infty} dx\, e^{ikx} \int_{0}^{\infty} dt\, e^{-st} w(x,z,t), \tag{2.4}$$

$$\hat{w}_0(k,z) = \int_{\infty}^{\infty} dx\, e^{ikx}\, w_0(x,z), \tag{2.5}$$

we arrive at the equation

$$\frac{d^2\hat{w}}{dz^2}+\left(\frac{N^2}{\Omega^2}+\frac{k^{-1}}{\Omega}\frac{d^2U}{dz^2}-1\right)k^2\hat{w} = k^2\left(\frac{g\hat{\rho}_0}{\bar{\rho}\Omega^2}+\frac{i\hat{w}_0}{\Omega}+\frac{1}{k\Omega}\frac{d\hat{u}_0}{dz}\right) = \hat{F}_0(z) \quad \text{(say)}, \tag{2.6}$$

where

$$N^2 = -g\frac{d\bar{\rho}}{dz}/\bar{\rho}$$

is the Brunt–Väisälä frequency squared and $\Omega = (-is-kU)$.

3. Idealized shear flow

Before considering the general problem it is instructive to examine the special case of unbounded, stratified, linear shear flow. Solutions can be obtained in terms of standard functions, and a comparison made with the previous studies of this idealized system. The results are also relevant to the problem of small-scale disturbances introduced into a flow that changes only over much larger scales.

If N^2 is a constant and the mean velocity field is a linear shear flow, so that

$$N^2 = B^2, \quad U = az,$$

(2.6) reduces to

$$\frac{d^2\hat{w}}{dy^2}+\left[\frac{B^2}{a^2(-y-is/a)^2}-1\right]\hat{w} = \left[\frac{g\hat{\rho}_0}{\bar{\rho}a^2(-y-is/a)^2}+\frac{i\hat{w}_0}{a(-y-is/a)}+\frac{1}{a(-y-is/a)}\frac{d\hat{u}_0}{dy}\right] = \hat{F}_0(y/k), \tag{3.1}$$

where $y = kz$. For definiteness we consider the case $a > 0$.

The solution of (3.1) may be written in terms of a Green's function involving the Bessel functions $I_{\pm\nu}$ and K_ν (see appendix). The conditions that $\hat{w}\to 0$ as $y\to\pm\infty$ imply that

$$\frac{d^2G(y,y')}{dy^2}+\left[\frac{B^2/a^2}{(y+is/a)^2}-1\right]G(y,y') = \delta(y-y') \tag{3.2}$$

with

$$\lim_{y\to\pm\infty} G(y,y') = 0.$$

It follows from the properties of $K_\nu(y)$ that

$$G(y,y') = -\frac{1}{\pi}\begin{cases}(y+is/a)^{\frac{1}{2}}K_\nu(y+is/a)\,(-y'-is/a)^{\frac{1}{2}}K_\nu(-y'-is/a) & \text{for } y > y', \quad (3.3a)\\ (\;y-is/a)^{\frac{1}{2}}K_\nu(-y-is/a)\,(y'+is/a)^{\frac{1}{2}}K_\nu(y'+is/a) & \text{for } y' > y, \quad (3.3b)\end{cases}$$

where

$$\nu^2 = \tfrac{1}{4}-B^2/a^2 \tag{3.4}$$

and B^2/a^2 is the Richardson number.

The solution of (3.1) is thus

$$\hat{w}(y) = \int_{\infty}^{\infty} dy'\,G(y,y')\,\hat{F}_0(y'/k) \tag{3.5}$$

and inverting the transforms gives

$$w(x,z,t) = \frac{1}{2\pi}\int_{\infty}^{\infty} dk\,e^{-ikx}\,\frac{1}{2\pi i}\int_{-i\infty+\gamma}^{i\infty+\gamma} ds\,e^{st}\int_{\infty}^{\infty} dy'\,G(y,y')\,\hat{F}_0(y'/k). \tag{3.6}$$

The form of w for large t

The asymptotic limit as $t \to \infty$ of w may be conveniently extracted from (3.6) by first performing the Laplace inversion. Both $G(y,y')$ and $\hat{F}_0(y'/k)$ are analytic functions in the right half of the complex s plane, so the inversion contour may be moved to within an infinitesimal distance ϵ of the imaginary s axis. Then as t becomes large the factor e^{st} in (3.6) becomes a rapidly oscillating function on the path of integration, and significant contributions to the integral arise only from those regions of s where other factors in the integrand have a correspondingly rapid variation. The only such regions are those about the zeros of $y'+is/a$ and $y+is/a$, where G and $\hat{F}_0$ have singularities, and the leading contribution (highest power of t) will come from the most singular of these.

The function $G(y,y')\hat{F}_0(y'/k)$ is replaced by its series expansion about the most singular region: the forms (3.1) of $\hat{F}$ and (3.3) of G show that the zero of $y'+is/a$ provides the highest singularity, and that the stratification term (with the explicit factor g) dominates.

Integrating the expanded form over s, the asymptotic series becomes (see appendix)

$$\lim_{t\to\infty}\frac{1}{2\pi i}\int_{-i\infty+\epsilon}^{i\infty+\epsilon} e^{st}G(y,y')\hat{F}_0(y'/k) = \frac{g\hat{\rho}_0(y'/k)}{\bar{\rho}(y'/k)}\, t^{\frac{1}{2}}\exp(iay't)\, C_5\{Q(y,y',\epsilon)$$
$$\times(C_1t^{\nu}+C_2t^{-\nu})+Q(-y,-y',-\epsilon)(C_3t^{\nu}+C_4t^{-\nu})\}+O(t^{-\frac{1}{2}\pm\nu}). \quad (3.7)$$

$C_1, \ldots, C_5$ are constants and

$$Q(y,y',\epsilon) = (y-y'+i\epsilon/a)^{\frac{1}{2}}K_\nu(y-y'+i\epsilon/a)\,\theta(y-y'), \quad (3.8)$$

where θ is the Heaviside function.

It is convenient to distinguish between systems for which $Ri \geqslant \frac{1}{4}$ and those for which $Ri < \frac{1}{4}$. In the former case ν is imaginary, while in the latter ν is real. The ensuing development will be confined to the case of ν real. The case ν imaginary may be treated in a parallel fashion; its behaviour is easily deduced from the formulae obtained with ν real.

The integration over y' is now treated using the same approach. The exponential factor ensures that at large t the significant region of the integrand is that around the zero of $y-y'+i\epsilon/a$, where $Q(y,y',\epsilon)$ varies rapidly. Expanding the functions Q and $\hat{\rho}/\bar{\rho}$ as power series about this point, and integrating over y', gives (see appendix)

$$\int_{-\infty}^{\infty} dy' \lim_{t\to\infty}\frac{1}{2\pi i}\int_{-i\infty+\gamma}^{i\infty+\gamma} ds\, e^{st}G(y,y')\hat{F}_0(y'/k)$$
$$= C\frac{g\hat{\rho}_0(y/k)}{\bar{\rho}(y/k)}\, e^{iayt}\, t^{2\nu-1}+O(t^{2\nu-2})+O(t^{-2\nu-1}). \quad (3.9)$$

This form gives the asymptotic behaviour of the perturbation in k space. Discussion of stability is usually made at this stage, rather than after a further inversion to x space, to avoid the complexities introduced by dispersion. However, the previous investigation by Case chose to examine stability in the x space representation. Then applying the k space inversion of (3.6) to the result (3.9) provides (remembering that $y = kz$)

$$\lim_{t\to\infty} w(x,z,t) = \frac{Cg}{\bar{\rho}(y/k)}\, t^{2\nu-1}\frac{1}{2\pi}\int_{-\infty}^{\infty} dk \exp[ik(azt-x)]\,\hat{\rho}_0(y/k). \quad (3.10)$$

Case (1960*b*) treats this inversion, which is slightly more complex in his formulation, as yet another asymptotic limit. He argues that, if the initial perturbation is of bounded extent in x, one has

$$\lim_{t\to\infty}\int dk\, e^{ikzat}\,\hat{\rho}_0(z,k) \sim t^{-1} \tag{3.11}$$

and hence

$$\lim_{t\to\infty} w(x,z,t) \sim t^{2\nu-2}\rho_0(x,z). \tag{3.12}$$

While this is correct mathematically, it is completely misleading physically. Returning to (3.10), this inversion can be obtained exactly:

$$\lim_{t\to\infty} w(x,z,t) = \frac{Cg}{\bar{\rho}(z)}\,t^{2\nu-1}\rho_0(x-zat,z), \tag{3.13}$$

or

$$\lim_{t\to\infty} w(x+Ut,z,t) \sim t^{2\nu-1}\rho_0(x,z). \tag{3.14}$$

The cause of the different time dependences in (3.12) and (3.14) is clear. The disturbance at height z is advected through the chosen frame of reference with the mean flow velocity $\hat{\mathbf{x}}U(z) = \hat{\mathbf{x}}az$. Consequently an observer at a fixed (x,z) point located in the region of the initial disturbance sees an additional 'decay' as the disturbance is advected away from him. Within the system as a whole, however, there is no associated fall-off; the amplitude may be recovered by observing the appropriate downstream location.

Nevertheless, since all real ν are less than 0·5 for finite shears, the time dependence in (3.14) corresponds to a decay in amplitude. This does not imply stability of the system, for consider the other scalar fields associated with the perturbation, namely (*a*) the horizontal velocity component u, (*b*) the density departure ρ_1, (*c*) the streamline displacement η and (*d*) the pressure field p_1.

(*a*) *The horizontal flow component.* The horizontal velocity field is obtained from the Fourier components of the field w through the incompressibility equation (2.2). For the Fourier component e^{-ikx},

$$u(k,z,t) = -ik^{-1}\,\partial w(k,z,t)/\partial z. \tag{3.15}$$

So from the k component of (3.10)

$$\lim_{t\to\infty} u(k,z,t) = agC\,\frac{\hat{\rho}_0(z)}{\bar{\rho}(z)}\,e^{ikzta}\,t^{2\nu} - \frac{i}{k}\,gC\left(\frac{d}{dz}\,\frac{\hat{\rho}_0(z)}{\bar{\rho}(z)}\right)e^{ikzta}\,t^{2\nu-1}. \tag{3.16}$$

The Fourier inversion follows as before, yielding the leading term

$$u(x+azt,z,t) \sim t^{2\nu}\rho_0(x,z)/\bar{\rho}(z). \tag{3.17}$$

The horizontal flow in the perturbation grows without limit.

(*b*), (*c*) *The density and streamline displacements.* These two functions are similar and may be treated together. The streamline displacement η obeys

$$D\eta/Dt = w \tag{3.18}$$

while the equation of mass conservation (2.3) gives

$$D\rho_1/Dt = -w\,d\bar{\rho}/dz. \tag{3.19}$$

The development of ρ_1 will be given.

The form (3.13) gives

$$\lim_{t\to\infty} D\rho_1/Dt \sim -(d\bar{\rho}/dz)\, t^{2\nu-1} \rho_0(x-azt, z), \tag{3.20}$$

from which

$$\lim_{t\to\infty} \rho_1(x,z,t) \sim -(d\bar{\rho}/dz)\, t^{2\nu} \rho_0(x-azt, z), \tag{3.21}$$

or

$$\lim_{t\to\infty} \rho_1(x+azt,z,t) \sim -(d\bar{\rho}/dz)\, \rho_0(x,z)\, t^{2\nu}, \tag{3.22}$$

so ρ_1 also grows without limit. The displacement η has the same behaviour.

It is interesting to check that the result (3.22) is also obtained through direct calculation using the asymptotic procedures. Applying the Laplace and Fourier transforms to (3.19) gives

$$\hat{\rho}_1 = -\left(\frac{d\bar{\rho}}{dz}\right)\frac{\hat{w}}{i\Omega} + \frac{\rho_0}{i\Omega}. \tag{3.23}$$

The Green's function formalism (3.5) for $\hat{w}$ is then applied, whence

$$\hat{\rho}_1 = -\left(\frac{d\bar{\rho}}{dz}\right)\frac{1}{i\Omega}\int_{-\infty}^{\infty} dy'\, G(y,y')\, \hat{F}_0(y') + \frac{\hat{\rho}_0}{i\Omega}. \tag{3.24}$$

Applying the inverse transforms gives

$$\rho_1(x,z,t) = \rho_0(x-at,z) + \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{-ikx} \times \frac{1}{2\pi i}\int_{i\infty+\gamma}^{i\infty+\gamma} ds\, e^{st}\left(\frac{-d\bar{\rho}}{dz}\right)\frac{1}{i\Omega}\int_{-\infty}^{\infty} dy' G(y,y')\, \hat{F}(y'). \tag{3.25}$$

The asymptotic evaluation of the triple integral proceeds exactly as in the development (3.6)–(3.14). The Laplace inversion gives a result that is essentially identical with (3.7). At the next step, namely the integration over the source co-ordinate y', the additional factor Ω in the denominator [the explicit Ω seen in (3.23)] comes into effect and contributes to the final time dependence $t^{2\nu}$.

(*d*) *The pressure field.* From the x component of (2.1) it is readily found that

$$p \sim t^{2\nu-1}, \tag{3.26}$$

and this field component decays at large times.

Detailed discussion of these results will be postponed until after examination of the general shear-flow problem, which shows essentially the same characteristics. At this point we simply note that the algebraic disturbances are a form of instability for flows with $Ri < \frac{1}{4}$.

4. The general shear-flow problem

The previous analysis depended entirely on the leading terms of the Frobenius expansions about the singularities of the governing equation. It should be apparent that this will also be true in the general problem where the background flow and the Brunt–Väisälä frequency are arbitrary functions of z. Also, it is known that the Taylor–Goldstein equation [the homogeneous form of (1.6)] yields essentially the same leading terms in a Frobenius expansion as does the equation obtained without the Boussinesq approximation; or for that matter the equation for subsonic perturbations in a compressible fluid. The only differences involve slowly varying multiplicative factors and, in the compressible system, a reformulation of N^2. Dispersion changes as the approximations are reduced, but we wish to examine stability in k space, where dispersion is not a consideration. Consequently (2.6) contains all the essential elements of more general formulations.

The boundary conditions for the problem are implicit in the Green's function. All those commonly used, viz. rigid upper or lower boundaries, outer regions where the solutions must obey a radiation or evanescence condition and orthodox combinations of these, can be included in a single formulation. The problem then is to find a solution of (2.6) subject to specification of the two general homogeneous boundary conditions

$$a_u \hat{w} + b_u d\hat{w}/dz = 0 \quad \text{at (or above) some upper value of } z, \tag{4.1}$$

$$a_l \hat{w} + b_l d\hat{w}/dz = 0 \quad \text{at (or below) some lower value of } z. \tag{4.2}$$

Let $\phi^u(k,z,s)$ and $\phi^l(k,z,s)$ be the solutions of the homogeneous form of (2.6) that respectively obey these upper and lower boundary conditions. The required solution is then

$$w(x,z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{-ikx}\, \frac{1}{2\pi i}\int_{-i\infty+\gamma}^{i\infty+\gamma} ds\, e^{st} \int_{-\infty}^{\infty} dz' G(z,z')\hat{F}_0(z'). \tag{4.3}$$

The Green's function

$$G(z,z') = \begin{cases} -\phi^u(z)\,\phi^l(z')/W & \text{for} \quad z > z', & (4.4a) \\ -\phi^u(z')\,\phi^l(z)/W & \text{for} \quad z < z', & (4.4b) \end{cases}$$

where W is the Wronskian of ϕ^u with ϕ^l.

We now examine the singularities of the product $G(z,z')\hat{F}_0(z')$ in the complex s plane. They are of two possible sorts. First, W may be zero for certain values s_i of s. This happens if, and only if, ϕ^u and ϕ^l are identical to within a multiplicative constant. In this case the homogeneous form of (2.6) plus the boundary conditions has an eigensolution ϕ_i with eigenvalue s_i. Such eigenvalues are expected to form a denumerable discrete set (which may be empty). Complex eigenvalues occur in complex-conjugate pairs, while the neutral modes have eigenvalues on the imaginary s axis. These zeros of W give regular singularities of G.

The second type of expected singularity comes from a zero of Ω. $\hat{F}_0$ has regular singularities, but the functions ϕ^u and ϕ^l have singularities of fractional order at such points. For any choice of k, z or z' there will exist one (pure imaginary) value of s for which Ω is zero. The functional form of $G(z,z')\hat{F}_0(z')$ near this value is found from the general Frobenius expansions as follows. Let z_c be a singular point of the equation

$$\frac{d^2\phi}{dz^2} + \left(\frac{N^2(z)}{\Omega^2} + \frac{k^{-1}}{\Omega}\frac{d^2U}{dz^2} - 1\right) k^2\phi = 0, \tag{4.5}$$

$$\Omega = -is - kU(z), \tag{4.6}$$

with

$$-is - kU(z_c) = 0. \tag{4.7}$$

The series expansion about the singularity is

$$\phi(z) = A(z-z_c)^{\frac{1}{2}-\nu}[1 + a_1(z-z_c) + \ldots] + B(z-z_c)^{\frac{1}{2}+\nu}[1 + b_1(z-z_c) + \ldots], \tag{4.8}$$

where

$$\nu^2 = \tfrac{1}{4} - N^2 \bigg/ \left[\frac{d^2U}{dz^2}\right]_{z=z_c}. \tag{4.9}$$

This is commonly used as a representation of ϕ over a small z domain about the singularity. However, by continuity it provides a representation over a small s domain with z fixed. The parameters $A, B, a_1, \ldots, b_1, \ldots, z_c$ and ν are then not constants, but continuous functions of the independent variable.

The locus of the singular point (4.7) may be inverted to give the s space relation

$$z_c = f(s). \tag{4.10}$$

The (fixed) field point z is a singularity for $s = \bar{s}(z)$, whence

$$z - z_c = f(\bar{s}) - f(s), \tag{4.11}$$

$$= -[\partial f/\partial s]_{\bar{s}}(s-\bar{s}) + \ldots, \tag{4.12}$$

$$\nu(s) = \nu(\bar{s}) + [\partial \nu/\partial s]_{\bar{s}}(s-\bar{s})\ldots, \tag{4.13}$$

$$A(s) = A(\bar{s}) + [\partial A/\partial s]_{\bar{s}}(s-\bar{s}) \tag{4.14}$$

and so forth.

Noting that

$$\begin{aligned}(s-\bar{s})^{\pm\nu} &= (s-\bar{s})^{\pm\alpha}, \quad \text{where} \quad \alpha = \nu(\bar{s}) + [\partial\nu/\partial s]_{\bar{s}}(s-\bar{s})\ldots \\ &= (s-\bar{s})^{\pm\nu(\bar{s})}\exp[\ln(s-\bar{s})[\{\pm[\partial\nu/\partial s]_{\bar{s}}(s-\bar{s})\ldots\}] \\ &= (s-\bar{s})^{\pm\nu(\bar{s})}\{1 \pm [\partial\nu/\partial s]_{\bar{s}}(s-\bar{s})\ln(s-\bar{s})\ldots\},\end{aligned} \tag{4.15}$$

the s space expansion of ϕ becomes

$$\begin{aligned}\phi(z,s) = {} & P(\bar{s})(s-\bar{s})^{\frac{1}{2}-\nu(\bar{s})}[1 + p_1(s-\bar{s}) + p_2(s-\bar{s}) + \ldots] \\ & + Q(\bar{s})(s-\bar{s})^{\frac{1}{2}+\nu(\bar{s})}[1 + q_1(s-\bar{s})\ln(s-\bar{s}) + q_2(s-\bar{s}) + \ldots].\end{aligned} \tag{4.16}$$

To define the fractional powers in s space a cut running along the imaginary s axis to $s = -i\infty$ is introduced at $s = \bar{s}$ ($\bar{s}$ is always pure imaginary). The location of all singularities and cuts is shown symbolically in figure 1.

Asymptotic form of the solution

The procedure used to obtain the long-time limit of (4.3) parallels that used previously for the idealized profile. The Laplace inversion is first performed. The s contour $-i\infty + \gamma \rightarrow i\infty + \gamma$ lies to the right of all singularities in the s plane, and is shown as the line Γ in figure 1. In the usual way, it may be displaced through the right-hand half-plane until it encounters the cut. Then the upper half of the contour may be swung through the left-hand half-plane until it lies against the other side of the cut. This results (figure 1*b*) in an infinite contour Γ' following the cut, plus a sequence of closed contours encircling the isolated singularities. The cut may be displaced an infinitesimal distance from the axis to avoid singularities such as s_k (figure 1) that lie there.

The isolated singularities arise from the existence of modes (eigensolutions) of the system, and the contours around them provide their excitation amplitudes. They are not the part of the solution of interest in this paper. The contour along the cut provides the algebraic disturbances, which we wish to examine. Thus we need to evaluate

$$I(z,z') = \frac{1}{2\pi i}\int_{\Gamma'} ds\, e^{st} G(z,z')\hat{F}_0(z'). \tag{4.17}$$

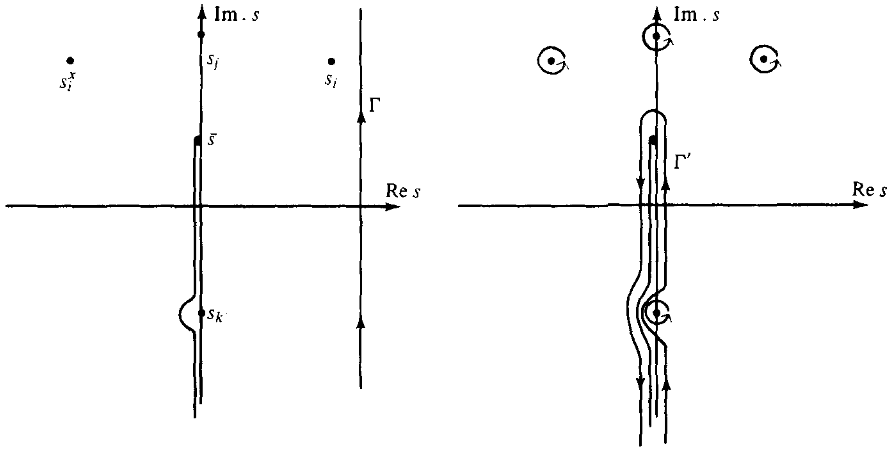

FIGURE 1. The s plane map. (a) Γ is the inversion contour for the Laplace transformation. s_i, s_i^*, s_j, s_k are eigenvalues of the flow system. $\bar{s}$ is the singular point that gives rise to the algebraic disturbances, and the origin of the cut. (b) Γ' is the component of the translated contour on which integration is performed to evaluate the algebraic disturbances.

In the limit $t \to \infty$ the exponential factor becomes very rapidly oscillating on Γ'. Significant contributions come only from the neighbourhood of singularities of the integrand, i.e. from the region about $\bar{s}$. The function $G(z, z')\,F(z')$ is replaced by a series expansion about $\bar{s}$. Since this is the only region that contributes in the asymptotic limit, the integration may be performed with this representation over the entire Γ' contour.

The integrations on Γ' are directly related to standard Fourier transformations:

$$\int_{\Gamma'} ds\, e^{st}(s-\bar{s})^{-p} \sim e^{\bar{s}t}\, t^{p-1}$$

and
$$\int_{\Gamma'} ds\, e^{st}\,(s-\bar{s})^{-p} \ln{(s-\bar{s})} \sim e^{\bar{s}t}\, t^{p-1}(A-\ln{(t)}).$$

In the asymptotic limit the leading term of (4.17) comes from the most singular term in the expansion:

$$\lim_{t\to\infty} I(z,z') = \frac{1}{2\pi i}\int_{\Gamma} ds\, e^{st}\,\frac{k^2}{W(k,\bar{s})}\,\frac{g\hat{\rho}_0(z')}{\bar{\rho}(z')\,(s-\bar{s})^2}\,\{\theta(z-z')\,\phi^u(k,z,\bar{s})\,P_l(\bar{s})\,(s-\bar{s})^{\frac{1}{2}-\nu(\bar{s})}$$
$$+\,\theta(z'-z)\,\phi^l(k,z,\bar{s})\,P_u(\bar{s})\,(s-\bar{s})^{\frac{1}{2}-\nu(\bar{s})}\} \quad (4.18)$$
$$= \frac{e^{\bar{s}t}\,k^2}{\pi W(k,\bar{s})}\,\frac{g\hat{\rho}_0(z')}{\bar{\rho}(z')}\,\Gamma(-\nu-\tfrac{1}{2})\sin{[\pi(\nu+\tfrac{3}{2})]}\,t^{\nu+\frac{1}{2}}\{\theta(z-z')\,\phi^u(k,z,\bar{s})\,P_l(\bar{s})$$
$$+\,\theta(z'-z)\,\phi^l(k,z,\bar{s})P_u(\bar{s})\}. \quad (4.19)$$

Here θ is the Heaviside function and P_u and P_l correspond to values of P in (4.16) for $\phi = \phi^u$ and $\phi = \phi^l$ respectively. The situation for $P_l(\bar{s})$ or $P_u(\bar{s})$ equal to zero will be examined later.

Now consider the z' integration in (4.3) applied to this asymptotic form $I(z,z')$. Both $\bar{s}$ and ν in (4.19) are functions of z': explicitly, $\bar{s} = ikU(z')$. The exponential factor $e^{\bar{s}t} = e^{ikU(z')t}$ is a rapidly oscillating function of z' as $t \to \infty$ provided that

$$dU(z')/dz' \neq 0.$$

(Any finite intervals of z' for which $dU(z')/dz' \equiv 0$ must be considered separately. They will not be discussed here although it should be apparent later that such regions will not support local algebraic instabilities.) As with the Laplace inversion, the rapidly oscillating exponential guarantees that integration over regions of z' where the rest of the integrand varies slowly contributes zero in the asymptotic limit. Only regions near singularities need be considered. Again, these may be identified as zeros of $W(k,\bar{s})$ and singularities of ϕ^u and ϕ^l. The former identify (neutral) modes of the homogeneous problem, and must be excluded from consideration because the previous contour of integration was deliberately distorted to exclude such points. This leaves the singularities of the functions ϕ^u and ϕ^l, which occur at (and only at) the critical levels identified by $i\bar{s}+kU(z) = 0$.

If U is a monotonic function of z', there is only one such singularity for any field point z. That is, when we held z and z' fixed to perform the s integration, the value of $U(z')$ determined $\bar{s}$, and no other point z' in the field can select the same value of $\bar{s}$. Now, the integration over z' must provide a value of $\bar{s}$ that makes z a critical level, and only one point on z', namely $z' = z$, can do this.

If U is not monotonic, however, there are (continua of) points z where the flow speed $U(z)$ is equal to the flow speed at other levels $z_1, z_2, \ldots \neq z$. Each of these levels gives the same value of $\bar{s}$, so when we integrate over z' the functions $\phi^{u,l}(z,\bar{s}(z'))$ will have singularities at $z' = z_1, z_2, \ldots$ as well as at $z' = z$.

We label all the points z_I that contribute a singularity to (4.19) as

$$\left.\begin{aligned} z' &= z = z_0, \\ z' &= (z_1, z_2, \ldots, z_N) > z, \\ z' &= (z_{-1}, z_{-2}, \ldots, z_{-M}) < z, \\ z_I &: U(z_I) = U(z). \end{aligned}\right\} \tag{4.20}$$

The integration over z' is most conveniently performed by making a transformation to the variable $\bar{s}$. The functions ϕ^u and ϕ^l are expanded [as in (4.10)–(4.16)] about each singular region z_I, after which each term is integrated over the entire domain. The procedure is essentially identical with the one previously used for the s space integration. The leading contribution from each singularity is selected, and we find

$$\begin{aligned} \lim_{t\to\infty} \int_{-\infty}^{\infty} dz' I(z,z') = \sum_{I=-M}^{N} 2gk^2 \frac{\exp[\frac{3}{2}i\pi(\nu(z)-\frac{1}{2})]\,\Gamma(\frac{3}{2}-\nu(z))}{\pi W(k,\bar{s}) \left| k\dfrac{dU_0}{dz}\right|_{z_I}} \Gamma(-\nu(z_I)-\tfrac{1}{2}) \\ \times \sin\pi(\nu(z)-\tfrac{1}{2}) \sin\pi(\nu(z_I)+\tfrac{3}{2}) \frac{\hat{\rho}_0(z_I)}{\bar{\rho}(z_I)} t^{\nu(z)+\nu(z_I)-1} \exp[ikU(z_I)t] \\ \times [\delta(I)\,P_u(k,z,\bar{s})\,P_l(k,z_I,\bar{s}) + \delta(-I)\,P_u(k,z_I,\bar{s})\,P_l(k,z,\bar{s})], \end{aligned} \tag{4.21}$$

where
$$\delta(q) = \begin{cases} 1 & \text{for} \quad q > 0, \\ \frac{1}{2} & \text{for} \quad q = 0, \\ 0 & \text{for} \quad q < 0 \end{cases}$$

and
$$\bar{s} = ikU(z).$$

5. Behaviour of the solution

Using (4.21) in (4.3) gives

$$\lim_{t\to\infty} w(x,z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \sum_{I=-M}^{N} \exp\left[ik(U(z_I)t-x\right]\hat{\rho}_0(z_I)\,C_1(k,z,\bar{s}) \times C_2(k,z_I,\bar{s})\,t^{\nu(z)+\nu(z_I)-1}. \quad (5.1)$$

The right-hand side of (5.1) is the first term of a series whose subsequent members have lower powers of t.

First consider the behaviour of a particular Fourier component k. Five essential points may be noted.

(i) In the long-time limit the disturbance at height z derives from initial disturbances at that height and from other heights z_I at which the mean flow speed matches that at z. No other regions of the fluid contribute to this leading term.

(ii) The response is frozen into the local fluid. That is, it is convected at the local fluid velocity $U(z_I) = U(z)$.

(iii) The time evolution of the disturbance can be written as the product of two algebraic terms, $t^{\nu(z)-\frac{1}{2}}$ and $t^{\nu(z_I)-\frac{1}{2}}$. The first depends only on the local properties at the field point z, the second only on the local properties at the contributing source point z_I of the initial disturbance. Each term depends explicitly on the stability of the flow as represented by the local Richardson number.

(iv) The amplitude factor $C_1(z)C_2(z_I)$ also displays the joint contributions of field-point and source-point properties. However, the functions C are not determined by the local properties of the system alone, but depend rather on the entire flow profile and the boundary conditions. The other terms on the right of (5.1) are indeed local, and require knowledge of only the initial disturbance and point values of U, N^2 and dU/dz. But it must not be concluded that, for instance, the second derivative of the velocity field plays no role in the problem. On the contrary, d^2U/dz^2 appears explicitly in the homogeneous form of (2.6); hence when ϕ^u and ϕ^l are traced from the boundaries through the fluid to the critical level, the second derivative of U plays a part in determining these functions, and hence the coefficients in the Frobenius expansions. These coefficients in turn determine C_1 and C_2. If we were interested in the magnitude of the perturbation at long times, rather than just the general information that the initial disturbance dies out or in some part grows without limit, we should have to specify the entire field and explicitly evaluate the C's.

(v) A factor $C(k,z,\bar{s})$ is zero if, and only if, the coefficient of the most singular term in the corresponding Frobenius expansion is zero. We expect that for each k component there may be a finite number of points in the z, $\bar{s}$ domain for which this could happen, but we do not anticipate continua of such points. However, the theory of fluid stability has a long history of providing the unexpected. So until it can be rigorously demonstrated otherwise, we must acknowledge the possibility that for some profiles C could be zero for a finite range of all the parameters.

Nevertheless, in the most commonly posed problem, a zero of C is isolated in the following sense. Consider the system in which the domain of interest is defined by rigid upper and lower plane boundaries. Suppose that for given values of $\bar{s}$ and k (and hence a particular value $z = z_c$ for the height of a singular point) the Frobenius expansion about z_c of the function ϕ^u is found to be

$$\phi^u(z-z_c) = B(z-z_c)^{\frac{1}{2}+\nu}(1+a_1(z-z_c)\ldots). \quad (5.2)$$

That is [see (4.8)], the series in $(z-z_c)^{\frac{1}{2}-\nu}$ is absent and $C(z_c, \bar{s}, k)$ is zero here. The analytic continuation of (5.2) to the upper boundary satisfies the condition

$$\phi^u(z = L) = 0.$$

Since we are dealing with a second-order differential equation, this is an isolated zero with respect to z. If we consider the contiguous problem in which the upper boundary is placed at $L \pm \delta$, the series (5.2) no longer satisfies the boundary condition. ϕ^u must then also contain a non-zero component of the series $(z-z_c)^{\frac{1}{2}-\nu}(1+\ldots)$ and $C(z_c, \bar{s}, k)$ is now non-zero.

Similarly, we could also consider contiguous problems in which slight variations of the functions U or N^2 eliminate a zero of C.

For these reasons we suspect that, if stability analysis is viewed as investigating types of flow characterized by, but not confined to, a particular realization, zeros of C are essentially isolated and expression (5.1) is valid.

Nature of the algebraic disturbances

Equation (5.1) closely resembles (3.14) and its treatment parallels that given in (3.14)–(3.26).

In a stably stratified medium, ν is real and less than $\frac{1}{2}$ where $Ri < \frac{1}{4}$, and is imaginary otherwise. Consequently

$$\lim_{t\to\infty} w(x,z,t) = 0. \quad (5.3)$$

Using the divergence equation (2.2) for the velocity, mass conservation (2.3) and the displacement relation (3.19), the following relations are established from (5.1):

$$\lim_{t\to\infty}\begin{Bmatrix} u(x,z,t) \\ \rho(x,z,t) \\ \eta(x,z,t) \end{Bmatrix} = \frac{1}{2\pi}\int dk \sum_{I\ M}^{N} \exp\left[ik(U(z_I)t-x\right]$$

$$\times \hat{\rho}_0(z_I)\, C_1(z)\, C_2(z_I)\, t^{\nu(z)+\nu(z_I)} \begin{cases} dU/dz, \\ (-)\,(d\bar{\rho}/dz)\,(\nu(z)+\nu(z_I))^{-1}, \\ (\nu(z)+\nu(z_I))^{-1}. \end{cases} \quad (5.4)$$

These fields all grow without limit if, and only if, the Richardson number at either the field point z or some contributing source point z_I is less than $\frac{1}{4}$. To the extent that our work is correct, this recovers Dyson's hypothesis regarding the stability of stratified flows. In a flow with a height-dependent Richardson number that in some intervals falls below the critical value of $\frac{1}{4}$, the fastest-growing algebraic disturbances will be those introduced at the location of the minimum of the Richardson number. Here the growth rate will be $t^{2\nu_{\max}}$, $\nu_{\max} = [(\frac{1}{4}-Ri)^{\frac{1}{2}}]_{\max}$.

The x component of Euler's equation (2.1) shows that the pressure field decays in the same manner as w.

If the flow contains a region that is unstable statically ($N^2 < 0$), ν is greater than $\frac{1}{2}$ there and all field components may increase without limit as $t \to \infty$. As such regions are known *a priori* to be unstable, the only information derived here is that the algebraic instabilities contribute to the evolution of the flow.

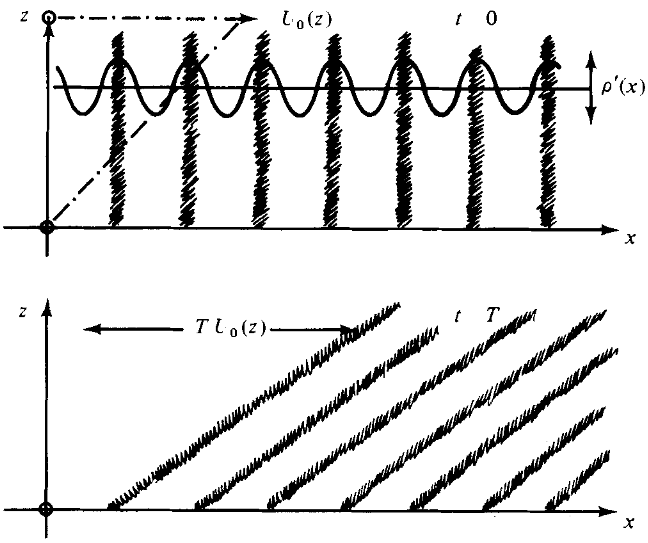

FIGURE 2. The mean flow profile $U_0(z)$ and its advective effect on the algebraic disturbances. At time $t = 0$ a density disturbance with sinusoidal variation in x and no structure on z is introduced. Heavy shading locates the density maxima. At time T later, advection has displaced the alignment, so that the density variation is now a strong function of z.

At first sight it seems strange that the various field components show different forms in the asymptotic limit, but this just reflects the nature of the singularity from which they are derived. It is known that in the modal analysis of the shear-flow problem the unstable modes must have a critical level (Howard 1961). At this level modes on the neutral-stability boundary have infinite horizontal flow speed, density perturbation and particle displacement, while the pressure and vertical flow fields are zero. This is precisely the asymptotic form found here for the non-modal components.

The conventional normal mode instabilities are wavelike, form-preserving perturbations. In principle this allows them to be readily identified in laboratory and field experiments. By contrast, the algebraic disturbances undergo continual distortion, and exhibit no strong k space selectivity to distinguish themselves from broad band background noise.

The distortion results from the height-dependent propagation term

$$\exp\left[ik(U(z)t-x)\right]$$

seen in (5.1) and (5.4). Since growth of the disturbance depends on the shear of $U(z)$, there is always vertical variation of the propagation (essentially an advection) rate. This is shown schematically in figure 2. At time $t = 0$ an initial disturbance is introduced into the shear flow. It is quasi-sinusoidal in the horizontal direction, and is confined and relatively free of structure in the z direction. After a time interval T, the differential advection has introduced a sinusoid-like variation in the z direction. From (5.3) and (5.4) it is seen that as $T \to \infty$ the local vertical wavelength tends to zero and the velocity field tends to a horizontal flow of infinite amplitude. Obviously this limit will never be achieved. Secondary instabilities and other nonlinear effects will come into play at earlier times. A quasi-horizontal flow with strong vertical shear should locally support conventional shear instabilities (Hazel 1972, especially § 4).

The lack of any marked k space selectivity in the algebraic system is a notable departure from the behaviour of modal instabilities. The latter assume scales that are determined by the extent of the strong shear regions (Miles & Howard 1964) or the spacing between the shear region and a solid boundary (Lalas, Einaudi & Fuá 1976; Lalas & Einaudi 1976). But the algebraic instabilities derive their scales only from the initial disturbances. In the nocturnal inversion of the earth's boundary layer, for instance, the modes are constrained to horizontal wavelengths of many hundreds of metres. Algebraic disturbances could exist with scales of (say) only a few metres.

This analysis of an isolated initial disturbance gives the impression that the algebraic instability should show up as a long narrow tilted ledge of activity. But in an active medium, such as the atmosphere, disturbances are more likely to be introduced continuously both in space and time. Thus, to the extent that 'initial' disturbances fill most of the fluid volume, the disturbances at later times will be found throughout regions of flow with mean $Ri < \frac{1}{4}$.

Although they result from linearized theory only, the algebraic disturbances behave more like turbulence than waves. In turbulence one expects convection with the local mean flow (Taylor's hypothesis), loss of form and a degeneration to smaller scales (cascading), and an absence of dominant scales imposed by the mean conditions.

6. Comparison with previous results; the completeness problem

We have already indicated [(3.12) *et seq.*] why our conclusions differ from those of Case. Essentially, Case formulated a wave-packet problem and examined the long-time behaviour of fields at the original location of the packet. Because, in general, the packet propagates (or advects) away from its initial position, the fields there tend to zero as $t \to \infty$, even if the amplitude of the packet grows with time. This calls to mind the famous admonition 'keep your eyes on the doughnut and not on the hole'.

Eliassen *et al.* examined various types of shear-flow problem. They concluded that in stably stratified systems with $Ri < \frac{1}{4}$ the algebraic disturbances always decay in the long-time limit. Moreover for some systems they identified a complete set of neutrally stable non-singular modes and concluded that in these cases there could consequently be no solutions of the algebraic type. Both of these points conflict with our findings and must be examined.

First we consider the decay/growth difference. Eliassen *et al.* do not explicitly formulate the initial-value problem, but work from an assumed representation (equations (6.1) and (6.2) in their paper). This can be compared with the Laplace approach through (4.3) *et seq.* It is very quickly found that their form is incorrect in that their amplitudes A_1 and A_2 are not, as assumed, functions of ζ (their notation) alone, but functions of both ζ and t. The reader may verify that

$$A_1 = -\frac{1}{2\pi i}\int_{-i\infty-\gamma}^{i\infty+\gamma} ds \exp\left[t(s+ik\zeta')\right]\frac{\phi^1(k,\zeta',s)}{\overline{W}(k,s)}\hat{F}_0(\zeta,s),$$

$$\lim_{t\to\infty} A_1 = t^{\nu+\frac{1}{2}}\hat{A}(\zeta) \quad \text{etc.}$$

Singular problems often display additional factors of t beyond the dependence associated with the non-singular system. The most familiar example is the simple harmonic oscillator driven at resonance. The Laplace formulation deals with such situations in a straightforward way.

The coexistence of algebraic disturbances with a complete set of neutral modes is also related to the singular nature of the system. In the examination of the s space properties of the Green's function [(4.3) *et seq.*], we identified isolated singularities corresponding to modes of the homogeneous equation, and the singularities of fractional order coming from the critical level of the equation. The latter give the algebraic disturbances. But suppose that the eigenmodes form a complete set. Then one might seek to express any initial disturbance in terms of this set, and there would appear to be no possible role for additional solutions. In fact, the original motivation for seeking algebraic disturbances came from the realization that certain systems, notably plane Couette flow, lacked a complete set of eigenmodes. Now, the Laplace formulation provides a solution of which the algebraic response is an essential part, without regard to eigenmode completeness or lack of it.

All semblance of contradiction disappears when it is recognized that completeness is always qualified by the order of continuity of the function space. The modes identified by Eliassen *et al.* (§ 3) are continuous, and have continuous first derivatives. The algebraic disturbances are discontinuous and have unbounded variation at the critical level. They are not representable as a series in the eigenmodes. Note, of course, that we are discussing a fixed component k of the Fourier representation, not the real x, z space functions, where physical considerations require boundedness. This reasoning is confirmed by comparing the results of Eliassen *et al.* with those of Booker & Bretherton (1967). The latter authors considered the fate of a wave train (periodic in x and t) propagating in a fluid with constant shear, constant stratification and strong stability ($Ri > \frac{1}{4}$). They found that in the vicinity of the critical level (where the horizontal phase speed matched the mean flow speed) the 'vertical wavelength' of the fields tended to zero; the horizontal velocity field, displacement and density perturbation tended to infinity, and the Reynolds stress was discontinuous. Their conclusion was that the greater part of the wave energy was being trapped in a vanishingly small height interval on the wave-incident side of the critical level. This system can be re-posed without significantly affecting the conclusions as follows.

Instead of a wave train, let the incident disturbance be a wave packet in k space whose amplitude at time $t = 0$ approaches zero at great distances above and below the critical-level interval. At arbitrarily greater distances insert rigid horizontal boundaries. The perturbation fields are evanescent in these outer regions (cf. the homogeneous form of (3.1) for $y \to \pm\infty$), so the boundaries are of no physical importance. The Booker & Bretherton work tells us that as $t \to \infty$ most of the initial wave packet will disappear into the critical-level regions. In particular, the fields of any k component will have all the properties deduced in their wave-train research.

But the system we have described, a constant shear flow with constant strong stability confined between two rigid boundaries, is identical with the one discussed in § 3 of Eliassen *et al.* There it was argued that the system has a complete set of well-behaved eigensolutions for any component. (The horizontal phase velocity of each of these solutions lies outside the limits of the background flow, so there are no critical levels.) It was concluded that: 'Assuming, without being able to prove it, that the set of eigenfunctions is complete, it follows that the system is [neutrally] stable... because the general solution may be interpreted as a superposition of [neutrally] stable waves.' We have inserted 'neutrally' here because the authors use the term stable to indicate the absence of instability, not, in this case, time decay.

We do not dispute the assumption of completeness in the conventional sense, relating to well-behaved continuous functions. But it is apparent that these eigen-solutions do not allow a description of the ill-behaved Booker & Bretherton wave train, any more than they include the algebraic disturbances.

7. Generalization to three space dimensions

The entire previous development has been confined to two space dimensions x and z. Generalization to the full three-dimensional space (x, y, z) follows exactly the procedure established for the standard modal theory of shear-flow instabilities.

Let

$$\mathbf{U}(z) = \hat{\mathbf{x}}U_x(z)+\hat{\mathbf{y}}U_y(z) \tag{7.1}$$

be the mean flow field in a three-dimensional system that is stratified in z. We follow closely the development and formalism of § 2 but take Fourier transformations of perturbation fields in two co-ordinates x and y. Thus in place of (2.4) we define

$$\hat{w}(k_x, k_y, z, s) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \exp[i(k_x x + k_y y)] \int_0^{\infty} dt\, e^{-st} w(x, y, z, t) \tag{7.2}$$

with similar generalizations of other Fourier transforms. This results in the following equation governing $\hat{w}$:

$$\frac{d^2\hat{w}}{dz^2} + \left[N^2k^2(\omega-\mathbf{k}\,.\,\mathbf{U})^{-2} + \frac{d^2(\mathbf{k}\,.\,\mathbf{U})}{dz^2}(\omega-\mathbf{k}\,.\,\mathbf{U})^{-1} - k^2\right]\hat{w}$$

$$= \left[g\rho_0 k^2\bar{\rho}^{-1}(\omega-\mathbf{k}\,.\,\mathbf{U})^{-2} + i\hat{\omega}_0 k^2(\omega-\mathbf{k}\,.\,\mathbf{U})^{-1} + (\omega-\mathbf{k}\,.\,\mathbf{U})\frac{d(\mathbf{k}\,.\,\mathbf{u}_0)}{dz}\right], \tag{7.3}$$

where

$$\mathbf{k} = \hat{\mathbf{x}}k_x+\hat{\mathbf{y}}k_y, \quad k^2 = \mathbf{k}\,.\,\mathbf{k}. \qquad (7.4), (7.5)$$

Define $U_{\mathbf{k}}$ and $u_{0,\mathbf{k}}$ through the projection operations

$$\mathbf{k}\,.\,\mathbf{U} = kU_{\mathbf{k}}, \quad \mathbf{k}\,.\,\mathbf{u}_0 = ku_{0,k}, \tag{7.6}$$

and observe that (7.3) is simply a particular form of (2.6) with (U, u_0) set equal to $(U_{\mathbf{k}}, u_{0,\mathbf{k}})$. This is no more than the physically expected result that the three-dimensional wave problem reduces precisely to the two-dimensional problem defined by the plane of propagation of the wave.

The stability properties for any $\mathbf{k}$ in the spectrum are now obtained from the results of §§ 2–6 with the identification of the relevant functions $U_{\mathbf{k}}$ and $u_{0,\mathbf{k}}$. Note in particular that the Richardson number must also be generalized,

$$Ri_{\mathbf{k}} = N^2(dU_{\mathbf{k}}/dz)^{-2}, \tag{7.7}$$

and that the growth or decay of a component of the initial disturbance depends strongly on its alignment with the vorticity of the local mean flow.

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Appendix. Algebraic disturbances in an idealized shear

Bessel functions of fractional order

$$\frac{d^2\phi}{dy^2}+\left[-1+\frac{\frac{1}{4}-\nu^2}{y^2}\right]\phi = 0 \tag{A 1}$$

is the canonical form of Bessel's equation of complex argument. Its solutions are

$$\phi = y^{-\frac{1}{2}} I_{\pm\nu}(y), \tag{A 2}$$

$$I_\nu = \sum_{m=0}^{\infty} \frac{(\frac{1}{2}y)^{\nu+2m}}{m!\,\Gamma(\nu+m+1)} \tag{A 3}$$

(Watson 1944, § 3.7).

In the limit $y \to \infty$ the $I_{\pm\nu}$ can be resolved into components with exponentially growing and exponentially decaying parts. In dealing with (3.1) we shall require the boundary condition that perturbations decay at large distances from source regions, so it is natural to use the function

$$K_\nu(y) = \tfrac{1}{2}\pi(I_\nu(y)-I_{-\nu}(y))/\sin \nu\pi \tag{A 4}$$

since

$$\lim_{y\to\infty} K_\nu(y) = (\pi/2y)^{\frac{1}{2}}\, e^{-y}. \tag{A 5}$$

From Watson [1944, § 3.71, result (18) and § 7.23, result (2)] it can be seen that there are no solutions of (A 1) that are bounded at both $y = \pm\infty$. Consequently the homogeneous form of (3.1) has no eigenmodes anywhere in the complex s plane. It is also useful to note that K_ν and I_ν require specification of the branch of y that is to be used. In this work it is always understood that this is done through a cut along the negative real axis of the complex y plane, so $-\pi < \arg y < \pi$. This ensures that in later manipulations the forms used are properly defined continuous solutions.

The Wronskian of $\{K_\nu(y), I_\nu(y)\}$ will be needed to set up the Green's functions. It is given in Watson [1944, § 3.71, result (19)].

Explicit evaluation of the solution in the idealized shear flow

In the main text we gave (3.6),

$$w(x,z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{-ikx}\, \frac{1}{2\pi i}\int_{i\infty+\gamma}^{i\infty+\gamma} ds\, e^{st} \int_{\infty}^{\infty} dy'\, G(y,y')\,\hat{F}_0(y'), \tag{A 6}$$

the Green's function (3.3),

$$G(y,y') = -\frac{1}{\pi}\begin{cases}(y+is_r/a)^{\frac{1}{2}}\,K_\nu(y+is_r/a)\,(-y'-is_r/a)^{\frac{1}{2}}\,K_\nu(-y'-is_r/a) & \text{for}\quad y > y', \qquad \text{(A 7a)}\\ (-y-is_r/a)^{\frac{1}{2}}\,K_\nu(-y-is_r/a)\,(y'+is_r/a)^{\frac{1}{2}}\,K_\nu(y'+is_r/a) & \text{for}\quad y' > y, \qquad \text{(A 7b)}\end{cases}$$

where

$$\nu^2 = \tfrac{1}{4}-B^2/a^2,$$

the source term (2.6),

$$\hat{F}_0(z) = k^2\left(\frac{g\hat{\rho}_0}{\bar{\rho}\Omega^2}+\frac{i\hat{w}_0}{\Omega}+\frac{1}{k\Omega}\frac{d\hat{u}_0}{dz}\right), \tag{A 8}$$

and indicated the steps required to obtain the final result (3.9). The details of these steps will now be given.

We begin with the s space integration. Equation (A 8) and the comment following (A 5) show that the functions are analytic in the right-hand half-plane, so the contour can be moved to be beside the imaginary s axis ($\gamma \to \epsilon$). Thus

$$
\begin{aligned}
I_1(y,y') &= \frac{1}{2\pi i}\int_{i\infty+\gamma}^{i\infty+\gamma} ds\, e^{st}\, G(y,y')\,\hat{F}_0(y) \\
&= \frac{i}{2\pi^2}\int_{-i\infty+\epsilon}^{i\infty+\epsilon} ds\, e^{st}\{\theta(y-y')\,(y+is/a)^{\frac{1}{2}}\,(-y'-is/a)^{\frac{1}{2}}\,K_\nu(y-is/a) \\
&\quad \times K_\nu(-y'-is/a)+\theta(y'-y)\,(-y-is/a)^{\frac{1}{2}}\,(y'+is/a)^{\frac{1}{2}}\,K_\nu(-y-is/a) \\
&\quad \times K_\nu(y'+is/a)\}\left\{\frac{g\hat{\rho}_0(y')}{\bar{\rho}(y')\,(y'+is/a)^2a^2}-\frac{i\hat{w}_0(y')}{(y'+is/a)\,a}+\frac{d\hat{u}_0(y')}{dy'}\right\},
\end{aligned}
\tag{A 9}
$$

where θ is the Heaviside function. As $t\to\infty$ the rapidly oscillating exponential ensures a contribution of zero from intervals where the rest of the integrand varies smoothly. Only those intervals about singularities of the function give something that survives in this limit. This is evaluated by expanding functions as their singular series and performing the integration for each term. Successive terms provide successively lower powers of t, so the asymptotic limit is obtained from the leading, most singular, term of the series. Thus

$$
\begin{aligned}
&\lim_{t\to\infty} I_1(y,y') \\
&= \frac{ig\hat{\rho}_0(y')}{4\pi a^2\bar{\rho}(y')\sin(\pi\nu)}\int_{-i\infty+\epsilon}^{i\infty+\epsilon} ds\, e^{st}\Big\{\theta(y-y')\,(y-y'+i\epsilon/a)^{\frac{1}{2}}\,K_\nu(y-y'+i\epsilon/a) \\
&\quad \times\left[\frac{(-y'-is/a)^{-\frac{1}{2}+\nu}}{2^\nu\Gamma(1+\nu)}-\frac{(y'-is/a)^{-\frac{1}{2}-\nu}}{2^{-\nu}\Gamma(1-\nu)}\right] \\
&\qquad\qquad + \text{same expression with } y\to -y,\, y'\to -y',\, i\to e^{i\pi}i\Big\} \\
&= \frac{-g\hat{\rho}_0(y')\exp(iy'at)}{2a^2\bar{\rho}(y')\sin(\pi\nu)}\Big\{\theta(y'-y)\,(y'-y-i\epsilon/a)^{\frac{1}{2}}\,K_\nu(y'-y-i\epsilon/a) \\
&\quad \times\left[\frac{t^{\frac{1}{2}-\nu}(i/a)^{-\frac{1}{2}+\nu}}{2^\nu\Gamma(1+\nu)\,\Gamma(\frac{3}{2}-\nu)}-\frac{t^{\frac{1}{2}+\nu}(1/a)^{-\frac{1}{2}-\nu}}{2^{-\nu}\Gamma(1-\nu)\,\Gamma(\frac{3}{2}+\nu)}\right] \\
&\qquad\qquad + \text{same expression with } y\to -y,\, y'\to -y',\, i\to e^{-i\pi}i\Big\},
\end{aligned}
\tag{A 10}
$$

which is the explicit expression represented by (3.7).

The next step is to integrate over y'. We assume weak stability ($Ri < \frac{1}{4}$), or static instability, so that ν is real and only the terms with $t^{\frac{1}{2}+\nu}$ need be carried. Then

$$
\begin{aligned}
\lim_{t\to\infty}\int_{-\infty}^{\infty} dy' I_1(y,y') &= \int_{-\infty}^{\infty} dy'\,\frac{t^{\frac{1}{2}+\nu}g\hat{\rho}_0(y')}{2^{1-\nu}a^2\bar{\rho}(y')\sin(\pi\nu)\,\Gamma(1-\nu)\,\Gamma(\frac{3}{2}+\nu)}\exp(iay't) \\
&\quad\times\{\theta(y'-y)\,(y'-y-i\epsilon/a)^{\frac{1}{2}}\,K_\nu(y'-y'-i\epsilon/a)\,a^{\frac{1}{2}+\nu} \\
&\quad + \text{same expression with } y\to -y,\, y'\to -y',\, i\to e^{-i\pi}i\}.
\end{aligned}
\tag{A 11}
$$

Again, the exponential factor reduces the integration to the regions about singularities. The only such interval is that around the zero of $y'-y$, and the leading term in the series representation gives

$$\lim_{t\to\infty}\int_{-\infty}^{\infty} dy' I(y,y') = \frac{t^{\frac{1}{2}+\nu} g\pi\hat{\rho}_0(y)\exp\left[\frac{1}{4}i\pi(2\nu-1)\right]}{\bar{\rho}(y)\, a^{\frac{1}{2}-\nu} 2^{2-\nu}\sin^2(\pi\nu)\,\Gamma(1-\nu)\,\Gamma(\frac{3}{2}+\nu)}$$

$$\times\int_{-\infty}^{\infty} dy' \exp(iay't)\,(y-y'+i\epsilon/a)^{\frac{1}{2}-\nu}$$

$$= C(\nu)\frac{g\hat{\rho}_0(y)}{\bar{\rho}(y)}\, t^{2\nu-1}\exp(iayt), \qquad \text{(A 12)}$$

which is the result presented in (3.9).

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